Filtering smooth concordance classes of topologically slice knots

* AMS Las Vegas *

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We are interested in studing knots modulo smooth concordance

Def$^{\underline{n}}$: The <u>knot concordance group</u> is

$$\mathcal{C} = \{\text{knots in } S^3\} \Big/ \begin{smallmatrix}\text{(smooth)} \\ \text{concordance}\end{smallmatrix}$$

In 1997, Cochran-Orr-Teichner defined the (n)-solvable filtration of $\mathcal{C}$ $(n \in \mathbb{N}/2)$

$$0 = \left\{ \begin{matrix} \text{slice} \\ \text{knots} \end{matrix} \right\} \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

- $\mathcal{F}_0$ = Arf invariant zero knots
- $\mathcal{F}_{0.5}$ = Algebraically slice knots
- $\mathcal{F}_{1.5} \subset$ knots with vanishing Casson-Gordon invariants.

Def A knot is <u>(n)-solvable</u>

M_K (0-surgery on K) bounds a

(1) $i_* : H_1(M_K) \xrightarrow{\cong} H_1(W)$

(2) $H_2(W)$ has a basis $\{f_i, g_i\}_{i=1}^{g}$ of embedded surfaces (w/ triv. normal bundle) all disjoint except $f_i \cdot g_i = 1$ (geometrically)

(3) $\pi_1(f_i), \pi_1(g_i) \subset \pi_1(W)^{(n)}$

- If $\pi_1(f_i) \subset \pi_1(W)^{(n+1)}$ as well then K is <u>(n.5) solvable</u>.

Def A knot $K \subset S^3$ is **n-solvable** if $\exists$ smooth 4-mfld W with $\partial W = S^3$ and $K = \partial \Delta$, where Δ is a smoothly embedded D^2 in W st. (1) $H_1(W) = 0$

(2) $\exists$ smoothly embedded surfaces l_i, d_i in $W \setminus \Delta$ with trivial normal bundles that are disjoint except $d_i \cap l_i = pt$ and $\{l_i, d_i\}_{i=1}^{g}$ is a basis for $H_2(W)$

(3) $\pi_1(l_i), \pi_1(d_i) \subset \pi_1^{(n)}(W - \Delta)$ (derived series of $\pi_1(W-\Delta)$)

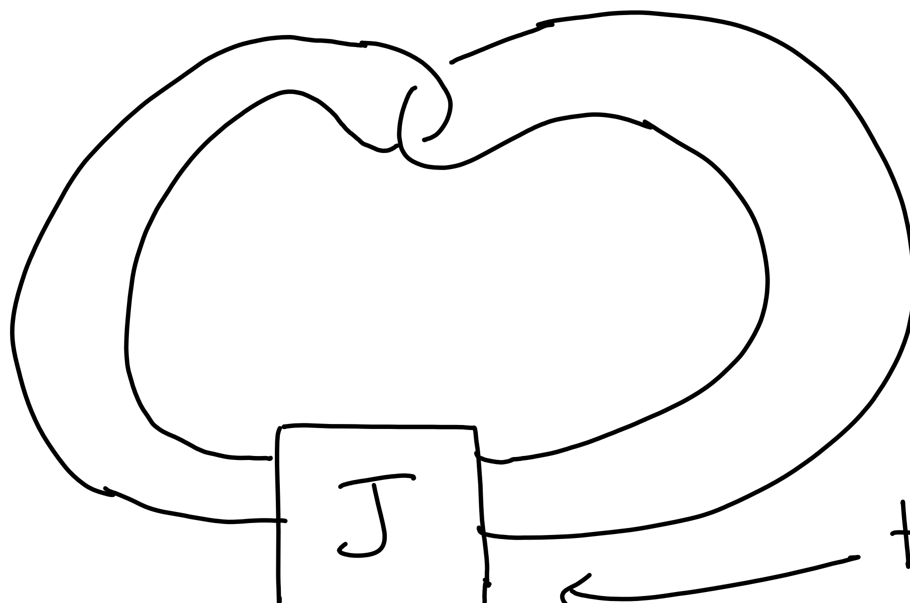
However, this filtration FAILS to distinguish a large class of knots called the topologically slice knots.

<u>Def</u> A knot K is topologically slice if $K = \partial D$ where D is a locally flat topological disk embedded in B^4.

Theorem (M. Freedman): If $\Delta_K(t) = 1$ (Alexander polynomial of K) then K is topologically slice

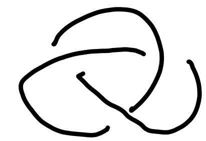
Ex: Whitehead double of J

$Wh(J) =$



tie band into knot J

- $\Delta_{Wh(J)} = 1$ so $Wh(J)$ is topologically slice.

- RHT = right handed trefoil = 

Known $Wh(RHT)$ is not (smoothly) slice.

$\Longrightarrow$ $Wh(RHT)$ is top but not smoothly slice.

Conjecture: $Wh(J)$ is smoothly slice $\Leftrightarrow$ J is smoothly slice

Denote by, $\mathcal{T}$, the (smooth) concordance classes of topologically slice knots.

$$\Rightarrow \{0\} \neq \mathcal{T} \subseteq \mathcal{C}$$

Endo showed that $\mathbb{Z}^\infty \subset \mathcal{T}$ and Hedden-Livingston-Ruberman showed that $\mathcal{T}/\Delta \supset \mathbb{Z}^\infty$ where Δ = knots smoothly concordant to knots w/ Alex. poly 1.

However, $\boxed{\Upsilon \subset \bigcap_{n=0}^{\infty} \mathcal{F}_n}$

We refine the n-solvable filtration to get a non-trivial filtration on Υ.

subgroup of topologically slice knots

smooth concordance group

$$0 \subset \cdots \subset \underbrace{\mathcal{T}_n \subset \cdots \subset \mathcal{T}_1 \subset \mathcal{T}_0}_{\text{new filtration}} \subset \mathcal{T} \subset \underbrace{\bigcap_{n=0}^{\infty} \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0}_{\text{COT filtration}} \subset \mathcal{C}$$

smoothly slice knots

Def A knot K is n-positive if $\exists$ a smooth 4-manifold W with $\partial W = S^3$ and $K = \partial D$, with $D \subset W$ a smoothly embedded disk s.t.

- $H_1(W) = 0$
- $\exists$ disjointly embedded surfaces $S_1, \dots, S_j$ with $S_i \cdot S_i = +1$ and $\{S_i\}$ is a basis for $H_2(W)$.
- $S_i \cap D = \phi \quad \forall\ i$
- $\pi_1(S_i) \subset \pi_1(W - D)^{(n)}$ (derived series) $\forall i$

Similar for n-negative but $S_i \cdot S_i = -1$.
(Euler class of normal bundle is -1).

Note: If K is smoothly slice $\Rightarrow$
$W = B^4$ and $H_2(W) = 0$ so K is
n-positive and n-negative for all n.

$P_n = \{\text{n-positive knots}\} \subset \mathcal{C}$

$N_n = \{\text{n-negative knots}\} \subset \mathcal{C}$

$NP_n = N_n \cap P_n$ is a filtration by subgroups of $\mathcal{C}$.

$$\cdots \subset NP_2 \subset NP_1 \subset NP_0 \subset \mathcal{C}$$

Thm (CHH): $NP_n \subset \mathcal{F}_n^{\text{no spin}}$ $\forall$ n. and

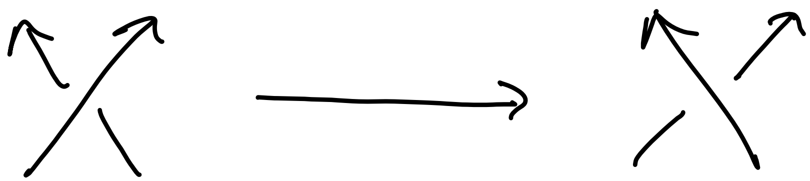
$$\bigoplus_{p(t)} \left(\mathbb{Z}^\infty \oplus \mathbb{Z}/2 \right) \subset NP_n / NP_{n+1} \quad \forall \ n.$$

Let $T_n = T \cap NP_n$ then

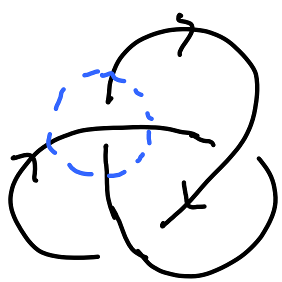
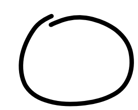
$$\cdots \subset T_1 \subset T_0 \subset T \subset C$$

We are interested in T_n / T_{n+1}.

<u>Prop</u> (CHH): If K can be changed to a slice knot by changing positive crossings

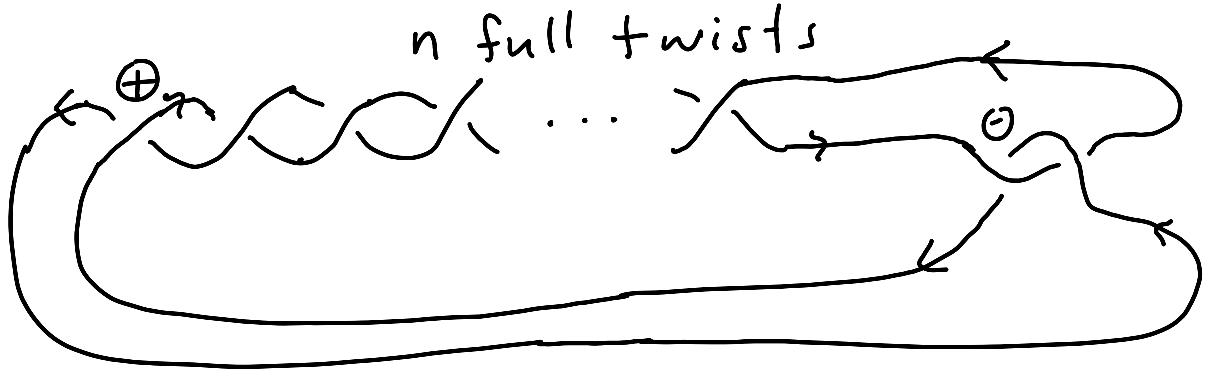


to negative crossings then $K \in P_0$ (similar for N_0).

<u>Ex</u>: RHT =  $\longrightarrow$  = 

$\Rightarrow$ RHT $\in P_0$

Ex: Twist knots

$Tw_n =$ 

Can change a positive or a negative crossing to get to the unknot

$$\Rightarrow Tw_n \in NP_0$$

Can show certain twist knots $\notin NP_1$.

Properties:

signature

(1) If $K \in P_0 \Rightarrow \sigma(K) \leq 0$ $\left(K \in N_0 \Rightarrow \sigma(K) \geq 0 \right)$

$\Rightarrow$ $\boxed{\text{If } K \in NP_0 \Rightarrow \sigma(K) = 0}$

Heegaard-Floer homology τ-invt

(2) If $K \in P_0 \Rightarrow \tau(K) \leq 0$ $\left(K \in N_0 \Rightarrow \tau(K) \leq 0 \right)$

$\boxed{\text{If } K \in NP_0 \Rightarrow \tau(K) = 0}$

Theorem (Cochran-H-Horn) Suppose $K \in P_1$ and Y is the p^r-fold cyclic branched cover of S^3 branched over K. There is a subgroup $G < H^2(Y)$ with $|G|^2 = |H^2(Y)|$ and a spin^c-structure $\mathfrak{s}$ on Y s.t. the Ozsváth-Szabó correction terms

$$d(Y, \mathfrak{s}+g) \leq 0$$

$\forall\ g \in G$.

Note: • $\mathfrak{s}$ = spin^c structure on Y that comes from a spin structure on Y and

$G \longleftrightarrow$ poincare dual of classes in $\ker(H_2(Y) \xrightarrow{i_*} H_2(W))$.

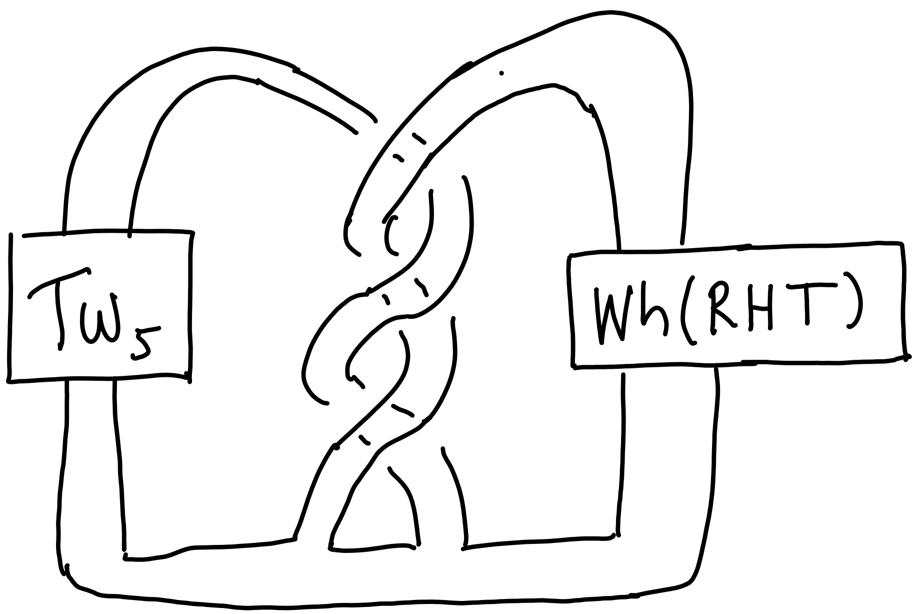
• If $K \in N_1$, get $d(Y, \mathfrak{s}+h) \geq 0$ for $h \in H$, some a priori different subgroup of $H^2(Y)$.

Using Casson-Gordon invariants and Ozsváth-Szabó d-invts we show:

Theorem (Cochran-H-Horn):

$$\boxed{\mathcal{T}_1/\mathcal{T}_2 \supset \mathbb{Z}}$$

Ex:



$\in \mathcal{T}_1$

$\notin \mathcal{T}_2$